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LETTER TO THE EDITOR

Some finite-size amplitudes and critical exponents for Potts and Ashkin–Teller quantum chains

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Abstract. Analytic expressions are derived for the leading finite-size corrections to a class of energy eigenvalues of the Potts and Ashkin–Teller quantum chains, making use of their equivalences with a modified *XXZ* Heisenberg chain. Assuming conformal invariance, exact results are thence obtained for some bulk scaling dimensions and surface exponents in these models.

In a recent paper (Hamer *et al* 1987, hereafter referred to as I), analytic expressions were derived for the finite-size scaling amplitude of the ground-state energy in the quantum Potts and Ashkin–Teller chains, by making use of their equivalences with a modified *XXZ* Heisenberg chain which can be solved by a Bethe ansatz (Alcaraz *et al* 1987a, b, c). By conformal invariance, these amplitudes are related to the conformal anomalies of their respective models. In the present work, we extend these results to derive the finite-size amplitudes for a class of excited states and the associated critical exponents.

The modified *XXZ* Hamiltonian in question is

$$H = -\tfrac{1}{2}\left(\sum_{j=1}^{N'} (\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z) + p\sigma_1^z + p'\sigma_N^z\right) \qquad (1)$$

where N is the number of sites, σ_i^x, σ_i^y and σ_i^z are Pauli matrices acting at site i, and $\Delta = -\cos\gamma$ where $\gamma \in [0, \pi)$. The cases of interest are as follows.

(A) $p = p' = 0$, $N' = N$, with boundary conditions

$$\sigma_{N+1}^x \pm i\sigma_{N+1}^y = e^{i\Phi}(\sigma_1^x \pm i\sigma_1^y) \qquad \sigma_{N+1}^z = \sigma_1^z. \qquad (2)$$

The eigenvalues of the critical q-state Potts Hamiltonian on an M-site lattice with periodic boundary conditions can be exactly related (Alcaraz *et al* 1987a, b) to those of chain A with $N = 2M$ sites, where $\cos\gamma = \frac{1}{2}\sqrt{q}$ and $\Phi = 2\gamma$.

(B) $N' = N - 1$, free boundaries. The eigenvalues of the critical q-state Potts chain on M sites with free boundaries are related (Alcaraz *et al* 1987a, b, c) to those of chain B with $\cos\gamma = \frac{1}{2}\sqrt{q}$, $N = 2M$, and $p = -p' = i\sin\gamma$. The eigenvalues of the critical Ashkin–Teller chain on M sites with free boundaries are also related to those of chain B with $N = 2M$, $p = p' = 0$, and the Ashkin–Teller coupling $\lambda = \cos\gamma$.

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The derivation of the finite-size amplitudes for the ground state in cases A and B in I made use of the methods of Woynarovich and Eckle (1987), and may be paraphrased as follows (the corresponding equations in I are given in round brackets). The total number of down spins m in the chain is conserved, and the ground state lies in the sector $m=\frac{1}{2}N$. The Bethe ansatz for the eigenstates involves a momentum p_j for each down spin, but in the critical region a convenient change of variables is (I, 2.5)

$$p=2\tan^{-1}[\cot(\tfrac{1}{2}\gamma)\tanh\lambda]\equiv\phi(\lambda,\tfrac{1}{2}\gamma) \qquad (-\infty<\lambda<\infty). \tag{3}$$

Then the Bethe ansatz equations for the roots λ_j corresponding to momenta p_j in cases A and B, respectively, may be written (Alcaraz *et al* 1987a, b, c)

$$\text{(A)}\ \ (\text{I}, 3.2)\quad N\phi(\lambda_j,\tfrac{1}{2}\gamma)=2\pi I_j+\Phi+\sum_{l=1}^{m}\phi(\lambda_j-\lambda_l,\gamma) \tag{4}$$

where

$$I_j=-[\tfrac{1}{2}(m+1)]+j \qquad j=1,\ldots,m \tag{5}$$

and

$$\text{(B)}\ \ (\text{I}, 3.3)\quad 2N\phi(\lambda_j,\tfrac{1}{2}\gamma)=2\pi I_j-\phi(\lambda_j,\Gamma)-\phi(\lambda_j,\Gamma')$$
$$+\sum_{\substack{l=1\\(\neq j)}}^{m}[\phi(\lambda_j-\lambda_l,\gamma)+\phi(\lambda_j+\lambda_l,\gamma)] \tag{6}$$

where

$$I_j=j \qquad j=1,\ldots,m \tag{7}$$

and

$$\text{(B)}\ \ (\text{I}, 3.4)\quad e^{2i\Gamma}=\frac{p-\Delta-e^{i\gamma}}{(p-\Delta)e^{i\gamma}-1} \qquad e^{2i\Gamma'}=\frac{p'-\Delta-e^{i\gamma}}{(p'-\Delta)e^{i\gamma}-1}. \tag{8}$$

A function $z_N(\lambda)$ can then be defined in which the roots are equally spaced, $z_N(\lambda_j)=I_j/N$, by

$$\text{(A)}\ \ (\text{I}, 3.7)\quad z_N(\lambda)=\frac{1}{2\pi}\left(\phi(\lambda,\tfrac{1}{2}\gamma)-\frac{\Phi}{N}-\frac{1}{N}\sum_{j=1}^{m}\phi(\lambda-\lambda_j,\gamma)\right) \tag{9}$$

$$\text{(B)}\ \ (\text{I}, 3.8)\quad z_N(\lambda)=\frac{1}{\pi}\bigg(\phi(\lambda,\tfrac{1}{2}\gamma)+\frac{1}{2N}[\phi(\lambda,\Gamma)+\phi(\lambda,\Gamma')+\phi(2\lambda,\gamma)]$$
$$-\frac{1}{2N}\sum_{j=1}^{m}[\phi(\lambda-\lambda_j,\gamma)+\phi(\lambda+\lambda_j,\gamma)]\bigg). \tag{10}$$

Its derivative is denoted

$$\sigma_N(\lambda)=dz_N(\lambda)/d\lambda. \tag{11}$$

When N goes to infinity, the roots λ_i tend to a continuous distribution with density $N\sigma_N(\lambda)$. Using the fact that

$$\int_{-\infty}^{\infty}\phi'(\lambda,\gamma)\,d\lambda=2(\pi-2\gamma) \tag{12}$$

(where the prime denotes differentiation with respect to λ), one obtains sum rules

$$\int_{-\infty}^{\infty}d\lambda\,\sigma_N(\lambda)=\begin{cases}\frac{1}{2} & \text{(A)}\ \ (\text{I}, 3.10) \qquad (13)\\ 1+(1/N)[3-2(\gamma+\Gamma+\Gamma')/\pi] & \text{(B)}\ \ (\text{I}, 3.11). \qquad (14)\end{cases}$$

Let $\Lambda_+(-\Lambda_-)$ denote the root of largest (smallest) magnitude in λ, then from (13) and (14) one finds in case A

$$\text{(A)}\quad \text{(I, 3.29)}\quad \begin{aligned} &\int_{\Lambda_+}^{\infty} \sigma_N(\lambda)\,\mathrm{d}\lambda = (1/2N)(1+\beta_+) \\ &\int_{-\infty}^{-\Lambda_-} \sigma_N(\lambda)\,\mathrm{d}\lambda = (1/2N)(1+\beta_-) \end{aligned} \tag{15}$$

where

$$\beta_\pm = \mp\Phi/\pi \tag{16}$$

while in case B $\Lambda_- = \Lambda_+ \equiv \Lambda$ and

$$\text{(B)}\quad \text{(I, 3.30)}\quad \int_{\Lambda}^{\infty} \sigma_N(\lambda)\,\mathrm{d}\lambda = (1/2N)(1+\beta) \tag{17}$$

where

$$\beta = 2[1-(\gamma+\Gamma+\Gamma')/\pi]. \tag{18}$$

The energy of the system is

$$\text{(A)}\quad \text{(I, 3.5)}\quad E = \tfrac{1}{2}N\cos\gamma - \sin\gamma \sum_{j=1}^{m} \phi'(\lambda_j, \tfrac{1}{2}\gamma) \tag{19}$$

$$\text{(B)}\quad \text{(I, 3.6)}\quad \mathrm{E} = \tfrac{1}{2}(\mathrm{N}-1)\cos\gamma - [\tfrac{1}{2}(p+p')] - \sin\gamma \sum_{j=1}^{m} \phi'(\lambda_j, \tfrac{1}{2}\gamma). \tag{20}$$

Using the Euler–Maclaurin formula and a Wiener–Hopf integration, as in Woynarovich and Eckle (1987), one can then go through to obtain the following results. Defining the energy per site as $e_N = E/N$, the leading finite-size correction to the ground-state energy per site as $N\to\infty$ is found to be

$$\text{(A)}\quad \text{(I, 3.37)}\quad e_N - e_\infty \approx -\frac{\pi^2 \sin\gamma}{6\gamma N^2}\left(1 - \frac{3\pi}{4(\pi-\gamma)}(\beta_+^2 + \beta_-^2)\right) \tag{21}$$

$$\text{(B)}\quad \text{(I, 3.38)}\quad e_N - e_\infty \approx -\frac{\pi^2 \sin\gamma}{24\gamma N^2}\left(1 - \frac{3\pi}{2(\pi-\gamma)}\beta^2\right) + \frac{f_\infty}{N} \tag{22}$$

where f_∞ is the surface energy in case B, discussed further in I. The next leading corrections are, for $\gamma \neq 0$,

$$\text{(A)}\quad \text{(I, 3.39)}\quad (1/N^2)[\mathrm{O}(N^{-2}) + \mathrm{O}(N^{-4\gamma/(\pi-\gamma)})] \tag{23}$$

$$\text{(B)}\quad \text{(I, 3.40)}\quad (1/N^2)[\mathrm{O}(N^{-1}) + \mathrm{O}(N^{-2\gamma/(\pi-\gamma)})] \tag{24}$$

while when $\gamma\to 0$ the next leading corrections are down by powers of $(\ln N)$.

We now want to consider an almost trivial generalisation of the above results. Consider the ground states in different spin sectors of the chain, with finite values of

$$n = \tfrac{1}{2}N - m. \tag{25}$$

Then equations (13) and (14) are replaced by

$$\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \sigma_N(\lambda) = \begin{cases} \tfrac{1}{2} + (n/N)(1-2\gamma/\pi) & \text{(A)} \quad (26) \\ 1 + (1/N)[3 - 2(\gamma+\Gamma+\Gamma')/\pi + 2n(1-2\gamma/\pi)] & \text{(B)} \quad (27) \end{cases}$$

and then the roots of largest magnitude are given by

$$\text{(A)}\qquad \begin{aligned}\int_{\Lambda_+}^{\infty}\sigma_N(\lambda)\,\mathrm{d}\lambda&=\frac{1}{2N}(1+\beta_+(n))\\ \int_{-\infty}^{-\Lambda_-}\sigma_N(\lambda)\,\mathrm{d}\lambda&=\frac{1}{2N}(1+\beta_-(n))\end{aligned}\tag{28}$$

where

$$\beta_\pm(n)=2n(1-\gamma/\pi)\mp\Phi/\pi\tag{29}$$

and

$$\text{(B)}\qquad \int_{\Lambda}^{\infty}\sigma_N(\lambda)\,\mathrm{d}\lambda=(1/2N)(1+\beta(n))\tag{30}$$

where

$$\beta(n)=4n(1-\gamma/\pi)+2[1-(\gamma+\Gamma+\Gamma')/\pi].\tag{31}$$

Following this, the treatment of I goes through unchanged, ending up with the leading finite-size correction to the ground-state energy per site in sector n as $N\to\infty$:

$$\text{(A)}\qquad e_N^{(n)}-e_\infty\simeq-\frac{\pi^2\sin\gamma}{6\gamma N^2}\left(1-\frac{3\pi}{4(\pi-\gamma)}(\beta_+(n)^2+\beta_-(n)^2)\right)\tag{32}$$

$$\text{(B)}\qquad e_N^{(n)}-e_\infty\simeq-\frac{\pi^2\sin\gamma}{24\gamma N^2}\left(1-\frac{3\pi}{2(\pi-\gamma)}\beta(n)^2\right)+\frac{f_\infty}{N}.\tag{33}$$

Note that the bulk limiting values e_∞ and f_∞ remain unchanged. The next leading corrections are of the same order as found above for $n=0$.

From equations (32) and (33) one easily obtains the mass gap between sector n and the overall ground state ($n=0$) as

$$F_N^{(n)}=N(e_N^{(n)}-e_N^{(0)})\tag{34}$$

with result

$$\text{(A)}\qquad F_N^{(n)}\underset{N\to\infty}{\sim}[\pi\sin\gamma/\gamma(\pi-\gamma)N][n^2(\pi-\gamma)^2+\tfrac{1}{4}\Phi^2]\tag{35}$$

$$\text{(B)}\qquad F_N^{(n)}\underset{N\to\infty}{\sim}(n\pi\sin\gamma/\gamma N)[n(\pi-\gamma)+\pi-(\gamma+\Gamma+\Gamma')].\tag{36}$$

Now the general form predicted for the mass gap by conformal invariance (Cardy 1984a) is

$$\text{(A)}\qquad F_N=2\pi\zeta x/N+\mathrm{O}(N^{-1})\tag{37}$$

for periodic boundary conditions, where x is the scaling dimension of the associated operator, and ζ is an overall scale factor (von Gehlen *et al* 1986) which is independent of the boundary conditions and is known (Hamer 1985) to be

$$\zeta=\pi\sin\gamma/\gamma\tag{38}$$

for the *XXZ* Hamiltonian (1). For the case of free boundaries, the corresponding relation is (Cardy 1984a)

$$\text{(B)}\qquad F_N=\pi\zeta x_s/N+\mathrm{O}(N^{-1})\tag{39}$$

where x_s is a surface scaling dimension. Assuming conformal invariance, then, one arrives at conclusions as follows.

For the Ashkin–Teller model with periodic boundaries, the scaling dimension of the 'spin-$\frac{1}{4}$ parafermion' operator has been shown to correspond to the gap $N(e_N^{(1)}(\Phi = \frac{1}{2}\pi) - e_N^{(0)}(\Phi = 0))$ in case A by Alcaraz *et al* (1987a, b). Hence one finds from equations (29) and (32):

$$x_{\rm pf}(\tfrac{1}{4}) = \frac{\pi-\gamma}{2\pi} + \frac{\pi}{32(\pi-\gamma)}. \tag{40}$$

This provides analytic confirmation of the numerical results of von Gehlen and Rittenberg (1987) and Alcaraz *et al* (1987a, b). Alcaraz *et al* (1987b) have also shown that the spin-$\frac{3}{4}$ parafermion operator corresponds to the gap $N(e_N^{(1)}(\Phi = 3\pi/2) - e_N^{(0)}(\Phi = 0))$. From (29) and (32) we find

$$x_{\rm pf}(\tfrac{3}{4}) = \frac{\pi-\gamma}{2\pi} + \frac{9\pi}{32(\pi-\gamma)} \tag{41}$$

confirming their result.

For the q-state Potts model with periodic boundaries, the scaling dimension of the magnetic operator corresponds to the gap $N(e_N^{(0)}(\Phi = \pi) - e_N^{(0)}(\Phi = 2\gamma))$ in case A, and hence one finds

$$x_\sigma = \frac{\pi^2 - 4\gamma^2}{8\pi(\pi-\gamma)} \tag{42}$$

with $\cos\gamma = \frac{1}{2}\sqrt{q}$. This confirms the numerical results of Alcaraz *et al* (1987a, b) and the identifications made by den Nijs (1983) and Dotsenko (1984). The Potts model also has parafermion operators (Fradkin and Kadanoff 1980, Nienhuis and Knops 1985) with spin $s = \alpha/q$ where $\alpha = 1, \ldots, q-1$. The corresponding mass gap has been shown by Alcaraz *et al* (1987b) to be $N(e_N^{(1)}(\Phi = 2\pi\alpha/q) - e_N^{(0)}(\Phi = 2\gamma))$ in case A, whence we find

$$x_{\rm pf}(\alpha; q) = \frac{\pi-\gamma}{2\pi} + \frac{\alpha^2\pi^2 - q^2\gamma^2}{2\pi q^2(\pi-\gamma)} \qquad \cos\gamma = \tfrac{1}{2}\sqrt{q} \tag{43}$$

confirming their conclusion and in agreement with the results of Nienhuis and Knops (1985).

For the case with free boundaries, case B, the gap $F_N^{(n)}$ is associated with a surface exponent $x_s^{(n)}$, which from (36), (38) and (39) is given by

$$x_s^{(n)} = n^2(\pi-\gamma)/\pi \tag{44}$$

for the XXZ model itself, and also for the Ashkin–Teller model, since $\Gamma = \Gamma' = \frac{1}{2}(\pi - \gamma)$ in both cases. This confirms the numerical results of Alcaraz *et al* (1987c) and von Gehlen and Rittenberg (1986, 1987). For the q-state Potts model, $\Gamma' = \pi - \Gamma$, the corresponding result is

$$x_s^{(1)} = (\pi - 2\gamma)/\pi \qquad \cos\gamma = \tfrac{1}{2}\sqrt{q} \tag{45}$$

in agreement with the numerical work of Alcaraz *et al* (1987c) and the prediction of Cardy (1984b).

To sum up, then, we have extended the work of I by deriving analytically the finite-size scaling amplitudes for the lowest-lying state in each spin sector of the modified XXZ Hamiltonian (1). Using conformal invariance, this has allowed us to calculate exactly various critical scaling dimensions for the quantum Potts and Ashkin–Teller chains. The results confirm previous numerical results and theoretical conjectures.

One would like to complete this work by calculating the finite-size scaling amplitudes corresponding to other excited states in the model. In general, such states will have complex roots in the λ plane, whose positions are not known exactly, although they might perhaps be calculated using methods such as those of Woynarovich (1982) (see also Woynarovich 1987). We have not attempted such lengthy calculations.

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